

INFINITESIMAL DEFORMATIONS OF A CALABI-YAU HYPERSURFACE OF THE MODULI SPACE OF STABLE VECTOR BUNDLES OVER A CURVE

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ABSTRACT. Let X be a compact connected Riemann surface of genus g , with $g \geq 2$, and \mathcal{M}_ξ a smooth moduli space of fixed determinant semistable vector bundles of rank n , with $n \geq 2$, over X . Take a smooth anticanonical divisor D on \mathcal{M}_ξ . So D is a Calabi-Yau variety. We compute the number of moduli of D , namely $\dim H^1(D, T_D)$, to be $3g - 4 + \dim H^0(\mathcal{M}_\xi, K_{\mathcal{M}_\xi}^{-1})$. Denote by \mathcal{N} the moduli space of all such pairs (X', D') , namely D' is a smooth anticanonical divisor on a smooth moduli space of semistable vector bundles over the Riemann surface X' . It turns out that the Kodaira-Spencer map from the tangent space to \mathcal{N} , at the point represented by the pair (X, D) , to $H^1(D, T_D)$ is an isomorphism. This is proved under the assumption that if $g = 2$, then $n \neq 2, 3$, and if $g = 3$, then $n \neq 2$.

1. INTRODUCTION

Let X be a compact connected Riemann surface of genus g , with $g \geq 2$. Let $\mathcal{M}_\xi := \mathcal{M}(n, \xi)$ denote the moduli space of stable vector bundles E of rank n , with $n \geq 2$, over X , such that the line bundle $\wedge^n E$ is isomorphic to a fixed holomorphic line bundle ξ over X . The degree $d = \deg(\xi)$ and n are assumed to be coprime. We also assume that if $g = 2$, then $n \neq 2, 3$, and if $g = 3$, then $n \neq 2$.

The moduli space \mathcal{M}_ξ is a connected smooth projective variety over \mathbb{C} , and for fixed n , the moduli space \mathcal{M}_ξ is isomorphic to $\mathcal{M}_{\xi'}$ if ξ' is another holomorphic line bundle with $\deg(\xi) = \deg(\xi')$. We take ξ to be of the form $L^{\otimes d}$, where L is a holomorphic line bundle over X such that $L^{\otimes (2g-2)}$ is isomorphic to the canonical line bundle K_X .

The Picard group $\text{Pic}(\mathcal{M}_\xi)$ is isomorphic to \mathbb{Z} . The anticanonical line bundle $K_{\mathcal{M}_\xi}^{-1}$ is isomorphic to $\Theta^{\otimes 2}$, where Θ is the ample generator of $\text{Pic}(\mathcal{M}_\xi)$, known as the *generalized theta line bundle*.

Let D be a smooth divisor on \mathcal{M}_ξ such that the holomorphic line bundle $\mathcal{O}_{\mathcal{M}_\xi}(D)$ over \mathcal{M}_ξ is isomorphic to $K_{\mathcal{M}_\xi}^{-1}$. Such a divisor is a connected simply connected smooth projective variety with trivial canonical line bundle. In other words, D is a Calabi-Yau variety.

If we move the triplet (X, L, D) , in the space of all triplets (X', L', D') , where D' is a smooth Calabi-Yau hypersurface on a moduli space of stable vector bundles, of the

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above type, over X' , then we get deformations of the complex manifold D , simply by associating the complex manifold D' to any triplet (X', L', D') . The Kodaira-Spencer infinitesimal deformation map for this family gives a homomorphism from the tangent space of the moduli space of triplets (X, L, D) , of the above type, into $H^1(D, T_D)$, the space parametrizing the infinitesimal deformations of the complex manifold D . The main result here, [Theorem 3.3], says

The above Kodaira-Spencer infinitesimal deformation map is an isomorphism.

Consequently, there is an exact sequence

$$(1.1) \quad 0 \longrightarrow \text{Hom}(l, H^0(\mathcal{M}_\xi, K_{\mathcal{M}_\xi}^{-1})/l) \longrightarrow H^1(D, T_D) \longrightarrow H^1(X, T_X) \longrightarrow 0,$$

where $l \subset H^0(\mathcal{M}_\xi, K_{\mathcal{M}_\xi}^{-1})$ is the one dimensional subspace defined by D . The above inclusion map

$$\text{Hom}(l, H^0(\mathcal{M}_\xi, K_{\mathcal{M}_\xi}^{-1})/l) \longrightarrow H^1(D, T_D)$$

corresponds to the deformations of D obtained by moving the hypersurface of the *fixed* variety \mathcal{M}_ξ , i.e., X is kept fixed, and the projection $H^1(D, T_D) \longrightarrow H^1(X, T_X)$ in (1.1) is the forgetful map from the space of infinitesimal deformations of the triplet (X, L, D) to the space of infinitesimal deformations of X . From the above exact sequence (1.1) it follows immediately that

$$\dim H^1(D, T_D) = 3g - 4 + \dim H^0(\mathcal{M}_\xi, K_{\mathcal{M}_\xi}^{-1}).$$

We note that the dimension of any $H^0(\mathcal{M}_\xi, \Theta^{\otimes k})$, in particular that of $H^0(\mathcal{M}_\xi, K_{\mathcal{M}_\xi}^{-1})$, is given by the Verlinde formula.

Let \mathcal{U}_D denote the restriction to $X \times D$ of a Poincaré vector bundle over $X \times \mathcal{M}_\xi$. For any $x \in X$, the vector bundle over D , obtained by restricting \mathcal{U}_D to $x \times D$, is denoted by $(\mathcal{U}_D)_x$. The following result is used in the proof of Theorem 3.3.

For any $x \in X$, the vector bundle $(\mathcal{U}_D)_x$ is stable with respect to any polarization on D . Moreover, the infinitesimal deformation map

$$T_x X \longrightarrow H^1(D, \text{Ad}((\mathcal{U}_D)_x)),$$

for the family \mathcal{U}_D of vector bundles over D parametrized by X , is an isomorphism.

This result was proved in [3, Theorem 2.5] under the assumption that $n \geq 3$. Here it is extended to the rank two case [Theorem 2.1].

2. RESTRICTION OF THE UNIVERSAL VECTOR BUNDLE

We continue with the notation of the introduction.

The anticanonical line bundle $K_{\mathcal{M}_\xi}^{-1} := \wedge^{\text{top}} T_{\mathcal{M}_\xi}$ is isomorphic to $\Theta^{\otimes 2}$ [6, page 69, Theorem 1], where the generalized theta line bundle Θ is the ample generator of the Picard group $\text{Pic}(\mathcal{M}_\xi)$; the Picard group is isomorphic to \mathbb{Z} .

Let $D \subset \mathcal{M}_\xi$ be a smooth divisor, satisfying the condition that the line bundle $\mathcal{O}_{\mathcal{M}_\xi}(D)$ is isomorphic to $K_{\mathcal{M}_\xi}^{-1}$. Let

$$\tau : D \longrightarrow \mathcal{M}_\xi$$

denote the inclusion map. Using the Poincaré adjunction formula, we have $K_D \cong \tau^* K_{\mathcal{M}_\xi} \otimes \tau^* \mathcal{O}_{\mathcal{M}_\xi}(D)$. In view of the assumption $\mathcal{O}_{\mathcal{M}_\xi}(D) \cong K_{\mathcal{M}_\xi}^{-1}$, the canonical line bundle K_D is trivial. Since the divisor D is ample, it is connected. Since the moduli space \mathcal{M}_ξ is simply connected, the divisor D is also simply connected. Therefore, D is a Calabi-Yau variety.

Fix a Poincaré vector bundle \mathcal{U} over $X \times \mathcal{M}_\xi$. In other words, for any $m \in \mathcal{M}_\xi$, the vector bundle over X obtained by restricting \mathcal{U} to $X \times m$ is represented by the point m . Let $\text{Ad}(\mathcal{U})$ denote the rank $n^2 - 1$ vector bundle over $X \times \mathcal{M}_\xi$ defined by the trace zero endomorphisms of \mathcal{U} . The vector bundle $(\text{Id}_X \times \tau)^* \mathcal{U}$ (respectively, $(\text{Id}_X \times \tau)^* \text{Ad}(\mathcal{U})$) over $X \times D$ will be denoted by \mathcal{U}_D (respectively, $\text{Ad}(\mathcal{U}_D)$).

For any fixed $x \in X$, let \mathcal{U}_x denote the vector bundle over \mathcal{M}_ξ obtained by restricting \mathcal{U} to $x \times \mathcal{M}_\xi$. The vector bundle over D obtained by restricting \mathcal{U}_D (respectively, $\text{Ad}(\mathcal{U}_D)$) to $x \times D$ will be denoted by $(\mathcal{U}_D)_x$ (respectively, $\text{Ad}(\mathcal{U}_D)_x$).

Since $H^2(D, \mathbb{Z}) = \mathbb{Z}$, the stability of a vector bundle over D does not depend on the choice of polarization needed to define the degree of a coherent sheaf over D .

Theorem 2.1. *For any point $x \in X$, the vector bundle $(\mathcal{U}_D)_x$ over D is stable. Moreover, the infinitesimal deformation map*

$$T_x X \longrightarrow H^1(D, \text{Ad}(\mathcal{U}_D)_x)$$

for the family \mathcal{U}_D of vector bundles over D parametrized by X , is an isomorphism.

Proof. If $n \geq 3$ and also $g \geq 3$, then the theorem has already been proved in [3, Theorem 2.5].

Take a point $x \in X$. We start, as in the proof of Theorem 2.5 of [3], by considering the exact sequence

$$0 \longrightarrow \text{Ad}(\mathcal{U})_x \otimes \mathcal{O}_{\mathcal{M}_\xi}(-D) \longrightarrow \text{Ad}(\mathcal{U})_x \xrightarrow{F} \tau_* \text{Ad}(\mathcal{U}_D)_x \longrightarrow 0,$$

over $X \times \mathcal{M}_\xi$, where F denotes the restriction map. This yields the long exact sequence

$$\begin{aligned} H^1(\mathcal{M}_\xi, \text{Ad}(\mathcal{U})_x \otimes \mathcal{O}_{\mathcal{M}_\xi}(-D)) &\longrightarrow H^1(\mathcal{M}_\xi, \text{Ad}(\mathcal{U})_x) \\ &\longrightarrow H^1(D, \text{Ad}(\mathcal{U}_D)_x) \longrightarrow H^2(\mathcal{M}_\xi, \text{Ad}(\mathcal{U})_x \otimes \mathcal{O}_{\mathcal{M}_\xi}(-D)) \end{aligned}$$

of cohomologies. If we consider \mathcal{U} as a family of vector bundles over \mathcal{M}_ξ parametrized by X , then the infinitesimal deformation map

$$T_x X \longrightarrow H^1(\mathcal{M}_\xi, \text{Ad}(\mathcal{U})_x)$$

is an isomorphism [5, page 392, Theorem 2]. In view of the above long exact sequence, to prove that the infinitesimal deformation map is surjective it suffices to establish the following lemma.

Lemma 2.2. *If $i = 1, 2$, then the following vanishing of cohomology*

$$H^i(\mathcal{M}_\xi, \text{Ad}(\mathcal{U}_x) \otimes \mathcal{O}_{\mathcal{M}_\xi}(-D)) = 0$$

is valid.

Proof of Lemma 2.2. This lemma was proved in [3, Lemma 2.1] under the assumption that $n \geq 3$. So in the proof we will assume that $n = 2$ and $g \geq 4$.

Let p denote, as in [3, Section 3], the natural projection of the projective bundle $\mathbb{P}(\mathcal{U}_x)$ over \mathcal{M}_ξ onto \mathcal{M}_ξ . Let T_p^{rel} denote the relative tangent bundle for the projection p from $\mathbb{P}(\mathcal{U}_x)$ to \mathcal{M}_ξ . Since $R^1 p_* T_p^{\text{rel}} = 0$ and $p_* T_p^{\text{rel}} \cong \text{Ad}(\mathcal{U}_x)$, for any $i = 0, 1, 2$, the isomorphism

$$H^i(\mathcal{M}_\xi, \text{Ad}(\mathcal{U}_x) \otimes \mathcal{O}_{\mathcal{M}_\xi}(-D)) = H^i(U, p^* K_{\mathcal{M}_\xi} \otimes T_p^{\text{rel}})$$

is obtained from the Leray spectral sequence for the map p .

If E is a stable vector bundle of rank two and degree one over X , then the vector bundle E' over X obtained by performing an elementary transformation

$$0 \longrightarrow E' \longrightarrow E \longrightarrow L_x \longrightarrow 0,$$

where L_x is a one dimensional quotient of the fiber E_x , is semistable. Therefore, we have a morphism, which we will denote by q , from $\mathbb{P}(\mathcal{U}_x)$ to the moduli space $\mathcal{M}_{\xi(-x)}$. Here $\xi(-x)$ denotes the line bundle $\xi \otimes \mathcal{O}_X(-x)$, and $\mathcal{M}_{\xi(-x)}$ is the moduli space of semistable vector bundles over X of rank two and determinant $\xi(-x)$.

Define $U \subset \mathbb{P}(\mathcal{U}_x)$ to be the inverse image, under that map q , of the stable locus of $\mathcal{M}_{\xi(-x)}$.

The line bundle T_p^{rel} is isomorphic to the relative canonical bundle K_q^{rel} [7, page 85], [5]. Therefore, to prove the lemma it suffices to show that

$$(2.3) \quad H^i(U, p^* K_{\mathcal{M}_\xi} \otimes K_q^{\text{rel}}) = 0,$$

where $i = 0, 1, 2$.

Using the isomorphism of T_p^{rel} with K_q^{rel} , from

$$(2.4) \quad q^* K_{\mathcal{M}_{\xi(-x)}} \otimes K_q^{\text{rel}} \cong K_U \cong p^* K_{\mathcal{M}_\xi} \otimes K_p^{\text{rel}}$$

we have

$$p^*K_{\mathcal{M}_\xi} \otimes K_q^{\text{rel}} \cong q^*K_{\mathcal{M}_{\xi(-x)}} \otimes \left(K_q^{\text{rel}}\right)^{\otimes 3}.$$

Since the restriction of the line bundle $p^*K_{\mathcal{M}_\xi} \otimes K_p^{\text{rel}}$ to a fiber of the map q has strictly negative degree, using the above isomorphism, and the projection formula, we have

$$(2.5) \quad H^i(U, p^*K_{\mathcal{M}_\xi} \otimes K_q^{\text{rel}}) = H^{i-1}\left(\mathcal{M}_{\xi(-x)}, K_{\mathcal{M}_{\xi(-x)}} \otimes R^1q_*\left(K_q^{\text{rel}}\right)^{\otimes 3}\right),$$

where $i = 0, 1, 2$.

The map q is smooth fibration \mathbb{CP}^1 fibration over an open subset U' of $\mathcal{M}_{\xi(-x)}$. The assumption that the genus of X at least four, ensures that the codimension of the complement of U' is at least four. Therefore, by using the Hartog type theorem for cohomology, the isomorphism (2.5) is established.

Setting $i = 0$ in (2.5), we conclude that $H^0(U, p^*K_{\mathcal{M}_\xi} \otimes K_q^{\text{rel}}) = 0$.

The following proposition is needed for our next step.

Proposition 2.6. *Let W be a holomorphic vector bundle of rank two over a complex manifold Z , and let $f : \mathbb{P}(V) \rightarrow Z$ be the corresponding projective bundle. Then there are canonical isomorphisms*

$$R^1f_*K_f^{\otimes 3} \cong S^4(W) \otimes \left(\bigwedge^2 W^*\right)^{\otimes 2} \cong R^0f_*T_f^{\otimes 2},$$

where K_f (respectively, T_f) is the relative canonical (respectively, anticanonical) line bundle.

Proof of Proposition 2.6. To construct the isomorphisms, let V be a complex vector space of dimension two. Choosing a basis of V , we identify the tangent bundle $T_{\mathbb{P}(V)}$ with $\mathcal{O}_{\mathbb{P}(V)}(2)$, and also obtain an identification of the line $\bigwedge^2 V^*$ with \mathbb{C} . Since,

$$H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(m)) = S^m(V),$$

we have an isomorphism of $H^0(\mathbb{P}(V), T_{\mathbb{P}(V)}^{\otimes 2})$ with $S^4(V) \otimes \left(\bigwedge^2 V^*\right)^{\otimes 2}$. Now it is a straight forward computation to check that this isomorphism is $GL(V)$ invariant, i.e., it does not depend on the choice of a basis of V . Therefore, this pointwise construction of a canonical isomorphism of vector spaces induces an isomorphism

$$R^0f_*T_f^{\otimes 2} \cong S^4(W) \otimes \left(\bigwedge^2 W^*\right)^{\otimes 2}$$

between vector bundles.

To obtain the other isomorphism in the statement of the proposition, first note that by the Serre duality we have $H^0(\mathbb{P}(V), T_{\mathbb{P}(V)}^{\otimes 2}) = H^1(\mathbb{P}(V), K_{\mathbb{P}(V)}^{\otimes 3})^*$. Now the canonical identification of $S^4(W) \otimes \left(\bigwedge^2 W^*\right)^{\otimes 2}$ with its dual, namely $S^4(W^*) \otimes \left(\bigwedge^2 W\right)^{\otimes 2}$, gives the other isomorphism. This completes the proof of Proposition 2.6. \square

The isomorphisms in Proposition 2.6 are *canonical isomorphisms*, i.e., they are compatible with the pull back of W using any map $Z' \rightarrow Z$, and furthermore, the isomorphisms are compatible with substituting W by $W \otimes L$, where L is a holomorphic line bundle over Z .

Combining Proposition 2.6 with (2.5), and using the projection formula, we get that if $i = 0, 1, 2$, then

$$(2.7) \quad \begin{aligned} H^i(U, p^* K_{\mathcal{M}_\xi} \otimes K_q^{\text{rel}}) &= H^{i-1} \left(\mathcal{M}_{(-x)}, K_{\mathcal{M}_{\xi(-x)}} \otimes q_* \left(T_q^{\text{rel}} \right)^{\otimes 2} \right) \\ &= H^{i-1} \left(U, q^* K_{\mathcal{M}_{\xi(-x)}} \otimes \left(T_q^{\text{rel}} \right)^{\otimes 2} \right). \end{aligned}$$

Indeed, the first isomorphism in (2.7) is a consequence of (2.5) and Proposition 2.6, and since $R^1 p_* \left(T_q^{\text{rel}} \right)^{\otimes 2} = 0$, the second isomorphism in (2.7) is valid. Although there is no universal vector bundle over $X \times \mathcal{M}_{\xi(-x)}$, the properties of the isomorphism $R^1 f_* K_f^{\otimes 3} \cong R^0 f_* T_f^{\otimes 2}$ in Proposition 2.6 that were explained earlier, evidently ensure that the isomorphism in (2.7) is valid. More precisely, the pointwise construction of the isomorphism between $R^1 q_* \left(K_q^{\text{rel}} \right)^{\otimes 3}$ and $q_* \left(T_q^{\text{rel}} \right)^{\otimes 2}$ gives an isomorphism of vector bundles.

Using (2.4), and the earlier mentioned fact that $T_p^{\text{rel}} \cong K_p^{\text{rel}}$, we obtain that

$$q^* K_{\mathcal{M}_{\xi(-x)}} \otimes \left(T_q^{\text{rel}} \right)^{\otimes 2} \cong p^* K_{\mathcal{M}_\xi} \otimes \left(K_p^{\text{rel}} \right)^{\otimes 3}.$$

Since the restriction of $\left(K_p^{\text{rel}} \right)^{\otimes 3}$ to a fiber of p has strictly negative degree, we have $p_* \left(K_p^{\text{rel}} \right)^{\otimes 3} = 0$. Consequently, the above isomorphism simplifies the terms in (2.7) to give the following isomorphism

$$(2.8) \quad H^{i-1} \left(U, q^* K_{\mathcal{M}_{\xi(-x)}} \otimes \left(T_q^{\text{rel}} \right)^{\otimes 2} \right) = H^{i-2} \left(\mathcal{M}_\xi, K_{\mathcal{M}_\xi} \otimes R^1 p_* \left(K_p^{\text{rel}} \right)^{\otimes 3} \right)$$

where $i = 0, 1, 2$.

Note that we obtain $H^1(U, p^* K_{\mathcal{M}_\xi} \otimes K_q^{\text{rel}}) = 0$ by setting $i = 1$ in (2.8).

In order to complete the proof of the lemma we need to show that

$$(2.9) \quad H^2(U, p^* K_{\mathcal{M}_\xi} \otimes K_q^{\text{rel}}) = 0.$$

To prove the above statement first observe that using (2.8), and setting $i = 2$, we have the following isomorphism

$$(2.10) \quad H^2(U, p^* K_{\mathcal{M}_\xi} \otimes K_q^{\text{rel}}) = H^0 \left(\mathcal{M}_\xi, K_{\mathcal{M}_\xi} \otimes R^1 p_* \left(K_p^{\text{rel}} \right)^{\otimes 3} \right).$$

Now using Proposition 2.6 we have

$$H^0\left(\mathcal{M}_\xi, K_{\mathcal{M}_\xi} \otimes R^1 p_* (K_p^{\text{rel}})^{\otimes 3}\right) = H^0\left(\mathcal{M}_\xi, K_{\mathcal{M}_\xi} \otimes S^4(\mathcal{U}_x) \otimes (\wedge^2 \mathcal{U}_x^*)^{\otimes 2}\right),$$

where \mathcal{U}_x , as defined earlier, is the vector bundle over \mathcal{M}_ξ obtained by restricting the Poincaré bundle \mathcal{U} to the subvariety $x \times \mathcal{U}_\xi \subset X \times \mathcal{U}_\xi$.

The vector bundle \mathcal{U}_x is known to be stable. Consequently, the vector bundle

$$S^4(\mathcal{U}_x) \otimes (\wedge^2 \mathcal{U}_x^*)^{\otimes 2}$$

is semistable. Now, since the vector bundle $S^4(\mathcal{U}_x) \otimes (\wedge^2 \mathcal{U}_x^*)^{\otimes 2}$ is the dual of itself, its degree is zero. On the other hand, the degree of $K_{\mathcal{M}_\xi}$ is strictly negative. From these it follows that the vector bundle

$$K_{\mathcal{M}_\xi} \otimes S^4(\mathcal{U}_x) \otimes (\wedge^2 \mathcal{U}_x^*)^{\otimes 2}$$

does not admit any nonzero section, since it is semistable of strictly negative degree. In view of (2.10), this establishes the assertion (2.9). Therefore, the assertion (2.3) is valid. This completes the proof of the lemma. \square

Since we have established, in Lemma 2.2, the rank two analog of Lemma 2.1 of [3], the proof of the stability of the vector bundle $(\mathcal{U}_D)_x$ for rank at least three, as given in [3, Theorem 2.5], is also valid for the rank two case if $g \geq 4$.

We note that [3, Theorem 2.5] was proved under the assumption that $g \geq 3$. However, the proof remains valid for $g = 2$ if the condition that the rank is at least four is imposed. Under this condition, the codimension of the subvariety over which the map q fails to be smooth and proper is sufficiently large in order to be able to apply the analog Hartog's theorem, which has been repeatedly used, for the cohomologies in question.

This completes the proof of Theorem 2.1. \square

In view of the above Lemma 2.2, all the results established in Section 2 of [3] for rank $n \geq 3$ remain valid for rank two and $g \geq 4$.

3. COMPUTATION OF THE INFINITESIMAL DEFORMATIONS

Let X be a compact connected Riemann surface of genus g , with $g \geq 2$. Take a holomorphic line bundle ξ over X of degree d . Let $\mathcal{M}_\xi := \mathcal{M}(n, \xi)$ denote the moduli space of stable vector bundles E of rank n over X , with $\wedge^n E = \xi$. For another line bundle ξ' of degree d , the variety $\mathcal{M}(n, \xi')$ is isomorphic to $\mathcal{M}(n, \xi)$. Indeed, if η is a line bundle over X with $\eta^{\otimes n} = \xi' \otimes \xi^*$, then the map defined by $E \mapsto E \otimes \eta$ is an isomorphism from $\mathcal{M}(n, \xi)$ to $\mathcal{M}(n, \xi')$. Therefore, we can rigidify (infinitesimally) the choice of ξ by the following procedure. Fix a line bundle L of degree one over X such that $L^{\otimes (2-2g)}$ is isomorphic to the tangent bundle T_X . We fix ξ to be $L^{\otimes d}$.

We will assume that the integers n and d are coprime, and $n \geq 2$. We will further assume that if $g = 2$ then $n \neq 2, 3$, and if $g = 3$, then $n \neq 2$.

The above numerical assumptions are made in order to ensure that the assertion in Theorem 2.1 is valid for \mathcal{M}_ξ .

Take a smooth divisor D on \mathcal{M}_ξ such that $\mathcal{O}_{\mathcal{M}_\xi}(D) = K_{\mathcal{M}_\xi}^{-1}$. Consider the exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{\mathcal{M}_\xi} \longrightarrow \mathcal{O}_{\mathcal{M}_\xi}(D) \longrightarrow \tau_* N_D \longrightarrow 0$$

over \mathcal{M}_ξ , where N_D is the normal bundle of the divisor D , and τ is the inclusion map of D into \mathcal{M}_ξ . Since

$$H^1(\mathcal{M}_\xi, \mathcal{O}_{\mathcal{M}_\xi}) = 0,$$

using the exact sequence of cohomologies, the space of sections $H^0(D, N_D)$ gets identified with the quotient vector space $H^0(\mathcal{M}_\xi, \mathcal{O}_{\mathcal{M}_\xi}(D))/\mathbb{C}$.

Let \mathcal{S} denote the space of all divisors D' on \mathcal{M}_ξ such that D' is homologous to D , i.e., they are represented by the same element in $H^2(\mathcal{M}_\xi, \mathbb{Z})$. Therefore, \mathcal{S} is identified with $\mathbb{P}H^0(\mathcal{M}_\xi, K_{\mathcal{M}_\xi}^{-1})$. The tangent space to \mathcal{S} , at the point $[D'] \in \mathcal{S}$ representing a divisor D' , has the following identification

$$T_{[D']}\mathcal{S} = H^0(D', N_{D'}) = H^0(\mathcal{M}_\xi, \mathcal{O}_{\mathcal{M}_\xi}(D'))/\mathbb{C}.$$

Let \mathcal{N} denote the moduli space of triplets of the form (X, L, D) , where X , L and D are as above (the line bundle L is a $(2g - 2)$ -th root of K_X). So \mathcal{N} is an open subset of moduli space of triplets of the form (X, L, α) , where α is a linear subspace of $H^0(\mathcal{M}_\xi, K_{\mathcal{M}_\xi}^{-1})$ of dimension one. The space \mathcal{N} parametrizes a family of Calabi-Yau varieties, simply by associating the Calabi-Yau variety D to any triplet $(X, L, D) \in \mathcal{N}$.

Take a point $\gamma := (X, L, D)$ in the moduli space \mathcal{N} . Associated to this family is the homomorphism

$$(3.1) \quad F : T_\gamma \mathcal{N} \longrightarrow H^1(D, T_D)$$

that maps the tangent space $T_\gamma \mathcal{N}$ of \mathcal{N} at γ to the space of infinitesimal deformation of the complex manifold D . In other words, this homomorphism F sends any tangent vector $v \in T_\gamma \mathcal{N}$ to the corresponding Kodaira-Spencer infinitesimal deformation class of D for the above family parametrized by \mathcal{N} .

The vector space $T_\gamma \mathcal{N}$ fits naturally into the short exact sequence

$$(3.2) \quad 0 \longrightarrow H^0(D, N_D) \longrightarrow T_\gamma \mathcal{N} \longrightarrow H^1(X, T_X) \longrightarrow 0,$$

where the projection $T_\gamma \mathcal{N} \longrightarrow H^1(X, T_X)$ corresponds to the forgetful map, which sends any point $(X', L', D') \in \mathcal{N}$ to the point represented by X' in the moduli space of Riemann surfaces; the inclusion $H^0(D, N_D) \longrightarrow T_\gamma \mathcal{N}$ in (3.2) corresponds to the obvious homomorphism $T_{[D]}\mathcal{S} \longrightarrow T_\gamma \mathcal{N}$, where \mathcal{S} , as before, is $\mathbb{P}H^0(\mathcal{M}_\xi, K_{\mathcal{M}_\xi}^{-1})$, the space of anticanonical divisors on \mathcal{M}_ξ .

Theorem 3.3. *The Kodaira-Spencer infinitesimal deformation map F constructed in (3.1) is an isomorphism of the tangent space $T_\gamma \mathcal{N}$ with $H^1(D, T_D)$.*

Proof. We start by considering the exact sequence

$$0 \longrightarrow T_D \longrightarrow \tau^* T_{\mathcal{M}_\xi} \longrightarrow N_D \longrightarrow 0$$

of vector bundles over D , where N_D is the normal bundle of D , and τ , as before, is the inclusion map of D into \mathcal{M}_ξ . This gives us the exact sequence

$$(3.4) \quad H^0(D, \tau^* T_{\mathcal{M}_\xi}) \longrightarrow H^0(D, N_D) \longrightarrow H^1(D, T_D) \longrightarrow H^1(D, \tau^* T_{\mathcal{M}_\xi}) \longrightarrow H^1(D, N_D)$$

of cohomologies.

Since the canonical line bundle K_D is trivial, and $N_D \cong \tau^* K_{\mathcal{M}_\xi}^{-1}$ is ample, the Kodaira vanishing theorem gives

$$(3.5) \quad H^1(D, N_D) = 0.$$

Therefore, the homomorphism $H^1(D, T_D) \longrightarrow H^1(D, \tau^* T_{\mathcal{M}_\xi})$ in (3.4) is surjective.

Our next aim is to show that

$$(3.6) \quad H^0(D, \tau^* T_{\mathcal{M}_\xi}) = 0,$$

which would be the first step in turning (3.4) into the short exact sequence (1.1) that we are seeking.

For that purpose, consider the vector bundle \mathcal{U}_D over $X \times D$ obtained by restricting a Poincaré bundle. Let ϕ (respectively, ψ) denote the projection of $X \times D$ to X (respectively, D). The vector bundle $R^1 \psi_* \text{Ad}(\mathcal{U}_D)$ over D is naturally isomorphic to $\tau^* T_{\mathcal{M}_\xi}$. Also, $\psi_* \text{Ad}(\mathcal{U}_D) = 0$, as the vector bundle $(\mathcal{U}_D)_x$ is stable, hence *simple*, for every $x \in X$ [Theorem 2.1]. The vector bundle $\text{Ad}(\mathcal{U}_D)$, as in Section 2, is the subbundle of $\text{End}(\mathcal{U}_D)$ consisting of trace zero endomorphisms. Now, using the Leray spectral sequence for the projection ψ , the isomorphism

$$H^0(D, \tau^* T_{\mathcal{M}_\xi}) = H^1(X \times D, \text{Ad}(\mathcal{U}_D))$$

is obtained.

The vector bundle $(\mathcal{U}_D)_x$ over D , defined in Section 2, has been proved to be stable in Theorem 2.1. So, we have $H^0(D, \text{Ad}((\mathcal{U}_D)_x)) = 0$ for every $x \in X$. Consequently, the isomorphism

$$H^1(X \times D, \text{Ad}(\mathcal{U}_D)) = H^0(X, R^1 \phi_* \text{Ad}(\mathcal{U}_D))$$

is obtained.

Now, from the second part of Theorem 2.1 we have a natural isomorphism

$$R^1 \phi_* \text{Ad}(\mathcal{V}_D) = T_X$$

obtained using the Poincaré bundle. Finally, since $H^0(X, T_X) = 0$, the assertion in (3.6) is an immediate consequence of the above isomorphism.

Using (3.5) and (3.6), the exact sequence in (3.4) reduces to

$$(3.7) \quad 0 \longrightarrow H^0(D, N_D) \longrightarrow H^1(D, T_D) \longrightarrow H^1(D, \tau^* T_{\mathcal{M}_\xi}) \longrightarrow 0.$$

The comparison of (3.7) with (3.2) shows that the next step has to be computation of $H^1(D, \tau^* T_{\mathcal{M}_\xi})$.

Consider the short exact sequence

$$0 \longrightarrow T_{\mathcal{M}_\xi} \otimes \mathcal{O}_{\mathcal{M}_\xi}(-D) \longrightarrow T_{\mathcal{M}_\xi} \longrightarrow \tau_* \tau^* T_{\mathcal{M}_\xi} \longrightarrow 0$$

of sheaves over \mathcal{M}_ξ . We know that $H^2(\mathcal{M}_\xi, T_{\mathcal{M}_\xi}) = 0$ [5, page 391, Theorem 1.a]. Also, we have (3.6). Consequently, the exact sequence yields the long exact sequence

$$(3.8) \quad \begin{aligned} 0 &\longrightarrow H^1(\mathcal{M}_\xi, T_{\mathcal{M}_\xi} \otimes K_{\mathcal{M}_\xi}) \longrightarrow H^1(\mathcal{M}_\xi, T_{\mathcal{M}_\xi}) \\ &\longrightarrow H^1(D, \tau^* T_{\mathcal{M}_\xi}) \longrightarrow H^2(\mathcal{M}_\xi, T_{\mathcal{M}_\xi} \otimes K_{\mathcal{M}_\xi}) \longrightarrow 0 \end{aligned}$$

of cohomologies; note that $H^i(\mathcal{M}_\xi, \tau_* \tau^* T_{\mathcal{M}_\xi}) = H^i(D, \tau^* T_{\mathcal{M}_\xi})$.

It was proved in [5] that the Kodaira-Spencer deformation map for \mathcal{M}_ξ , as the Riemann surface X moves in the moduli space of Riemann surfaces, is an isomorphism of $H^1(\mathcal{M}_\xi, T_{\mathcal{M}_\xi})$ with $H^1(X, T_X)$. Therefore, comparing (3.2) with (3.7), and using the exact sequence (3.8), we conclude that in order to complete the proof of the theorem, it suffices to establish the following statement : if $i = 1, 2$, then

$$(3.9) \quad H^i(\mathcal{M}_\xi, T_{\mathcal{M}_\xi} \otimes K_{\mathcal{M}_\xi}) = 0.$$

Indeed, (3.9) implies that $H^1(D, \tau^* T_{\mathcal{M}_\xi}) = H^1(\mathcal{M}_\xi, T_{\mathcal{M}_\xi}) = H^1(X, T_X)$.

To prove (3.9), let δ denote the dimension of the variety \mathcal{M}_ξ . The Serre duality gives the following isomorphism

$$(3.10) \quad H^i(\mathcal{M}_\xi, T_{\mathcal{M}_\xi} \otimes K_{\mathcal{M}_\xi}) = H^{\delta-i}(\mathcal{M}_\xi, \Omega_{\mathcal{M}_\xi}^1)^* = H^{1, \delta-i}(\mathcal{M}_\xi)^*.$$

(Here $H^{j,k}(\mathcal{M}_\xi) := H^k(\mathcal{M}_\xi, \Omega_{\mathcal{M}_\xi}^j)$.)

To finish the proof of the statement (3.9) we need to use some properties of the Hodge structure of the cohomology algebra $H^*(\mathcal{M}_\xi, \mathbb{C})$, which will be recalled now.

Fix a Poincaré bundle \mathcal{U} over $X \times \mathcal{M}_\xi$. Let $c_k := c_k(\mathcal{U}) \in H^{k,k}(X \times \mathcal{M}_\xi)$ denote the k -th Chern class of \mathcal{U} . For any $\alpha \in H^{i,j}(X)$, we have

$$(3.11) \quad \lambda(k, \alpha) := \int_X c_k \cup f^* \alpha \in H^{k+i-1, k+j-1}(\mathcal{M}_\xi),$$

where f denotes the obvious projection of $X \times \mathcal{M}_\xi$ onto X , and \int_X is the Gysin map for this projection, which is constructed by integrating differential forms on $X \times \mathcal{M}_\xi$ along the fibers of the projection f . The collection of all these cohomology classes

$\{\lambda(k, \alpha)\}$, constructed in (3.11), generate the cohomology algebra $H^*(\mathcal{M}_\xi, \mathbb{C})$ [1, page 581, Theorem 9.11]. On the other hand, we know that the following

$$H^{0,1}(\mathcal{M}_\xi) = 0$$

is valid.

With these properties of $H^*(\mathcal{M}_\xi, \mathbb{C})$ at our disposal, we are in a position to prove that the algebra generated by the cohomology classes $\{\lambda(k, \alpha)\}$ cannot have a nonzero element in $H^{1,\delta-1}(\mathcal{M}_\xi)$ or $H^{1,\delta-2}(\mathcal{M}_\xi)$, where $\delta = \dim_{\mathbb{C}} \mathcal{M}_\xi$.

To prove the above assertion, suppose that

$$\omega = \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_l$$

is a nonzero element in $H^{1,\delta-1}(\mathcal{M}_\xi) \oplus H^{1,\delta-2}(\mathcal{M}_\xi)$, where $\omega_j \in \{\lambda(k, \alpha)\}$ for all $j \in [1, l]$. We will see that $\omega_j \in H^{0,1}(\mathcal{M}_\xi)$ for at least one $j \in [1, l]$. Since $H^{0,1}(\mathcal{M}_\xi) = 0$, this would prove that $\omega = 0$.

First observe that in (3.11), we have $k+i-1 \geq k-1$ and $k+j-1 \leq k$, as $\dim_{\mathbb{C}} X = 1$. In other words, we have

$$(3.12) \quad (k+j-1) - (k+i-1) \leq 1.$$

Let $\omega_i \in H^{a_i, b_i}(\mathcal{M}_\xi)$, where $i \in [1, l]$. Then $a_i \leq 1$, and consequently from (3.12) the inequality $b_i \leq 2$ is obtained. Furthermore, $a_j \neq 0$ for at most one $j \in [1, l]$. If $a_i = 0$, then $b_i \leq 1$; but the possibility $b_i = 1$ is ruled out as $H^{0,1}(\mathcal{M}_\xi) = 0$. Therefore, all ω_i except one is a scalar. Now, if $a_j = 1$, then from (3.12) we have $b_j \leq 2$. On the other hand, we have $\delta - 2 > 2 \geq b_j$. Consequently, we conclude that $\omega = 0$.

Since the cohomology classes $\lambda(k, \alpha)$ are of pure type, i.e.,

$$\lambda(k, \alpha) \in H^{a,b}(\mathcal{M}_\xi)$$

for some integers a and b , it is easy to see that for any $i \geq 0$, the cohomology group $H^i(\mathcal{M}_\xi, \mathbb{C})$ is generated, as a complex vector space, by completely decomposable elements, i.e., elements of the type ω considered above. Therefore, we have $H^{1,\delta-1}(\mathcal{M}_\xi) = 0 = H^{1,\delta-2}(\mathcal{M}_\xi)$.

In view of (3.10), this completes the proof of the statement (3.9). We already noted that the statement (3.9) completes the proof of the theorem. \square

As a consequence of Theorem 3.3 we get that

$$\dim H^1(D, T_D) = 3g - 4 + \dim H^0(\mathcal{M}_\xi, \Theta^{\otimes 2}).$$

The dimension of $H^0(\mathcal{M}_\xi, \Theta^{\otimes 2})$ is given by the Verlinde formula.

Remark 3.13. From [2, page 760, Proposition 1], coupled with [2, page 759, Théorème 1], it follows that $H^0(D, T_D) = 0$. We note that this is also an immediate consequence of (3.6).

We have $H^2(D, T_D) = H^{m-1,2}(D) = H^{m,3}(\mathcal{M}_\xi)$, where $m = \dim D$; the second isomorphism is obtained from the Lefschetz hyperplane theorem [4, page 156]. The earlier proof that $H^{1,\delta-i}(\mathcal{M}_\xi) = 0$ for $i = 1, 2$, easily extends to prove that $H^{1,\delta-3}(\mathcal{M}_\xi) = 0$. Therefore, we have $H^2(D, T_D) = 0$. However, by a theorem due to Bogomolov-Kawamata-Tian-Todorov it is already known that the deformations of a Calabi-Yau variety are unobstructed.

REFERENCES

- [1] M. F. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Philos. Trans. Roy. Soc. Lond. A **308** (1982), 523–615.
- [2] A. Beauville, *Variétés Kähleriennes dont la première classe de Chern est nulle*, Jour. Diff. Geom. **18** (1983), 755–782.
- [3] I. Biswas and L. Brambila-Paz, *Restriction of the Poincaré bundle to a Calabi-Yau hypersurface*, to appear in Jour. reine angew. Math. (e-print: math.AG/9902145).
- [4] P. Griffiths and J. Harris, *Principles of algebraic geometry*, A Wiley-Interscience publication, 1978.
- [5] M. S. Narasimhan and S. Ramanan, *Deformations of the moduli space of vector bundles over an algebraic curve*, Ann. of Math. **101** (1975), 391–417.
- [6] S. Ramanan, *The moduli spaces of vector bundles over an algebraic curve*, Math. Ann. **200** (1973), 69–84.
- [7] A. N. Tyurin, *The geometry of moduli of vector bundles*, Russ. Math. Surveys **29**:6 (1974), 57–88.

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